Necessary and sufficient condition for hydrostatic equilibrium in general relativity

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Abstract

We present explicit examples to show that the 'compatibility criterion' [recently obtained by us towards providing equilibrium configurations compatible with the structure of general relativity] which states that: for a given value of $\sigma[\equiv (P_0/E_0) \equiv$ the ratio of central pressure to central energy-density], the compactness ratio $u[\equiv (M/R)$, where M is the total mass and R is the radius of the configuration] of any static configuration cannot exceed the compactness ratio, u_h, of the homogeneous density sphere (that is, $u \leq u_h$), is capable of providing a *necessary* and *sufficient* condition for any regular configuration to be compatible with the state of hydrostatic equilibrium. This conclusion is drawn on the basis of the finding that the $M - R$ relation gives the necessary and sufficient condition for dynamical stability of equilibrium configurations only when the compatibility criterion for these configurations is appropriately satisfied. In this regard, we construct an appropriate sequence composed of core-envelope models on the basis of compatibility criterion, such that each member of this sequence satisfies the extreme case of causality condition $v = c = 1$ at the centre. The maximum stable value of $u \simeq 0.3389$ (which occurs for the model corresponding to the maximum value of mass in the mass-radius relation) and the corresponding central value of the local adiabatic index, $(\Gamma_1)_0 \simeq 2.5911$, of this model are found fully consistent with those of the corresponding *absolute* values, $u_{\max} \leq 0.3406$, and $(\Gamma_1)_0 \leq 2.5946$, which impose strong constraints on these parameters of such models.

In addition to this example, we also study dynamical stability of pure adiabatic polytropic configurations on the basis of variational method for the choice of the 'trial function', $\xi = re^{\nu/4}$, as well as the mass-central density relation, since the compatibility criterion is appropriately satisfied for these models. The results of this example provide additional proof in favour of the statement regarding compatibility criterion mentioned above. Together with other results, this study also confirms the previous claim that just the choice of the 'trial function', $\xi = re^{\nu/4}$, is capable of providing the necessary and sufficient condition for dynamical stability of a mass on the basis of variational method. Obviously, the upper bound on compactness ratio of neutron stars, $u \cong 0.3389$, which belongs to two-density model studied here, turns out to be much stronger than the corresponding 'absolute' upper bound mentioned in the literature.

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1. INTRODUCTION

Einstein's field equations for static and spherically symmetric mass distribution were first solved by Schwarzschild (1916). The first solution describes the geometry of the space-time exterior to a prefect fluid sphere in hydrostatic equilibrium. While the other, known as interior Schwarzschild solution, corresponds to the interior geometry of a fluid sphere of constant (homogeneous) energy-density, E. The importance of these two solutions in general relativity is well known. The interior Schwarzschild solution provides two very important features towards obtaining configurations in hydrostatic equilibrium, compatible with general relativity, namely - (i) It gives an absolute upper limit on compactness ratio, $u (\equiv M/R$, mass to size ratio of the entire configuration in geometrized units$) \leq (4/9)$ for any static and spherical configuration (belonging to arbitrary density profiles, provided the density does not increase outwards) in hydrostatic equilibrium (Buchdahl 1959; Weinberg 1972), and (ii) For an assigned value of the compactness ratio, u, and radius R (or mass M), the minimum central pressure, P_0, corresponds to the homogeneous density solution (see, e.g., Weinberg 1972).

Despite the non linear and coupled differential equations, various exact solutions of the field equations for static and spherically symmetric metric are available in the literature (see, e. g., Kramer et al 1980) which may be used to obtain various physical properties of spherical and static compact object (provided they are physically realistic). Knutsen (1988; 1989) examined physical properties of the various exact solutions and found that these solutions correspond to nice physical properties and also remain stable against small radial pulsations upto certain values of u.

Another way to explore the physical properties of compact objects like neutron star, one may expect to have some physically viable equation of state (EOS). However, for such objects the equations of state (EOSs) are not well known [empirically] because of the lack of knowledge of nuclear interactions beyond the density $\sim 10^{14} \mathrm{g\,cm^{-3}}$ (Dolan 1992), and the only way to obtain EOSs far beyond this density range is extrapolation. Various such

extrapolated equations are available in the literature (Arnett & Bowers 1977). As a way out, one can impose some restrictions upon the known physical quantities, such that, the speed of sound inside the configuration, $v[\equiv \sqrt{(dP/dE)}]$, does not exceed the speed of light in vacuum, i.e., $v \leq c = 1$ (in geometrized units), and obtain various physical properties, like upper bound on stable neutron star masses (Rhoades & Ruffini 1974; Brecher & Caporaso 1976; Hartle 1978; Friedman & Ipser 1987). Haensel and Zdunik (1989) have shown that the only EOS which can describe a submillisecond pulsar and the static mass of $1.442 M_\odot$ simultaneously, corresponds to the EOS, $(dP/dE) = 1$, however, they emphasized that this EOS represents an 'abnormal' state of matter in the sense that pressure vanishes at densities of the order of nuclear density or even higher.

We have recently proposed a core-envelope model with stiffest EOS, $(dP/dE) = 1$, forming the core and a polytropic equation with constant adiabatic index $\Gamma_1 = (\mathrm{d}lnP/\mathrm{d}ln\rho)$ [where P is the pressure and ρ represents the rest-mass density] describing the envelope, such that P, E, both of the metric parameters (ν, and λ), their first derivatives, and the speed of sound are continuous at the core-envelope boundary and at the exterior boundary (surface) of the structure (Negi & Durgapal 2000). The other remarkable feature of this core-envelope model is that not only the 'abnormalities' (in the sense discussed in the literature, see, e.g. Lee 1975; Haensel & Zdunik 1989) disappear, the maximum value of $u \cong 0.3574$ for the stable configuration turns out to be as large as that obtained by using the EOS, $(dP/dE) = 1$, alone (see, e. g., Haensel & Zdunik 1989). The maximum stable mass of neutron star based upon this model (by using the maximum value of $u \cong 0.3574$ for stable configuration) turns out to be $7.944 M_\odot$, if the (average) density of the configuration is constrained by fastest rotating pulsar, with rotation period, $P_{rot} \cong 1.558$ ms, known to date. The model gives pulsationally stable configurations with compactness ratio $u > (1/3)$, which are important to study Ultra-Compact Objects (UCOs) [see, e. g., Negi & Durgapal 1999a, b; 2000; and references therein].

Recently, by using property (ii) of the homogeneous density sphere as mentioned above, we have connected the compactness ratio, u, of any static and spherical configuration with

the corresponding ratio of central pressure to central energy-density $\sigma[\equiv (P_0/E_0)]$ and worked out an important criterion which concludes that for a given value of σ, the maximum value of compactness ratio, $u(\equiv u_h)$, should always correspond to the homogeneous density sphere (Negi & Durgapal 2001).

An examination of this criterion on some well known exact solutions and EOSs indicated that this criterion, in fact, is fulfilled only by two types of configurations corresponding to a *single* EOS or density variation: (i) the *regular* (positive finite density at the origin which decreases monotonically outwards) configurations which correspond to a vanishing density at the surface together with pressure [so called, the gravitationally-bound structures] (Negi & Durgapal 2001; Negi 2004b; Negi 2006), and (ii) the structures which correspond to a non-vanishing surface density but exhibit singularities at the centre, in the sense that both pressure and density become infinity as $r \to 0$ [so called, the self-bound singular structures] (Negi 2004b; Negi 2006). On the other hand, it is seen that the EOSs or analytic solutions, corresponding to a non-zero *finite*, surface density (that is, the pressure vanishes at finite surface density, and so called the self-bound regular structures), in fact, do not fulfill this criterion (Negi & Durgapal 2001; Negi 2004b; Negi 2006). We have shown this inconsistency particularly for the EOS, $(dP/dE) = 1$ (as it represents the most successful EOS to obtain the various extreme characteristics of neutron stars as discussed above).

In addition to the self-bound regular configurations corresponding to a single density variation, the compatibility criterion may not be satisfied by various two-density, or multiple-density, regular, gravitationally-bound structures. Such structures are widely discussed in the literature, particularly, for determining the upper bound on neutron star (NS) masses. In this connection, we would consider the core-envelope model proposed by Negi & Durgapal (2000).

The reason(s) behind non-fulfillment of the criterion obtained in the study of Negi and Durgapal (2001) by various exact self-bound regular solutions and EOSs, as well as the two-density, gravitationally-bound, regular structures, and their further implications are discussed in the following sections.

2. Compatibility criterion: the necessary and sufficient condition for hydrostatic equilibrium of any static spherical configuration

In order to bring things together regarding the compatibility of regular structures mentioned above, we follow Negi and Durgapal (2001) by assuming a homogeneous sphere of uniform energy-density, E. The equations for isotropic pressure P, and uniform energy-density E, can be written in terms of compactness ratio, u, and the radial coordinate measured in units of configuration size, $y(\equiv r/R)$ as

$$8\pi E R^2 = 6u. \tag{1}$$

$$8\pi P R^2 = 6u\Big[\frac{(1 - 2uy^2)^{1/2} - (1 - 2u)^{1/2}}{3(1 - 2u)^{1/2} - (1 - 2uy^2)^{1/2}}\Big]. \tag{2}$$

Let us consider a *regular* variable density sphere (with some given EOS or analytic solution) with central energy-density E_0 and central pressure P_0 , corresponding to the compactness ratio $u = u_v$.

Now, we can always construct a homogeneous density sphere with the same value of the compactness ratio u_v, and energy-density E_0, because if P_{0h} corresponds to the central pressure of this sphere, the ratio $\sigma_h(\equiv P_{0h}/E_0)$ depends only upon the assigned value of the compactness ratio u_v . And, P_{0h} is given by

$$P_{0h} = (6u/8\pi R^2)\Big[(1 - (1 - 2u)^{1/2})/(3(1 - 2u)^{1/2} - 1)\Big]. \tag{3}$$

Now, according to property (ii) of homogeneous density sphere, we may write

$$P_0 \geq P_{0h} \tag{4}$$

or,

$$(P_0/E_0) \geq (P_{0h}/E_0). \tag{5}$$

Hence for a given value of $u(\equiv u_v)$, we obtain

$$\sigma_v \geq \sigma_h \tag{6}$$

where σ_v is defined as the ratio, (P_0/E_0).

Now, varying the compactness ratio, u_v, for the homogeneous density sphere from u_v to u_h (say), such that, we should have

$$\sigma_v = \sigma_h. \tag{7}$$

For $u = u_h$, the value of σ_h would become

$$\sigma_h = \left[((1 - 2u_h)^{1/2} - 1)/(1 - 3(1 - 2u_h)^{1/2})\right]. \tag{8}$$

Substituting Eq. (8) with the help of Eq. (7) into Eq. (6), we get

$$\left[\frac{(1 - 2u_h)^{1/2} - 1)}{(1 - 3(1 - 2u_h)^{1/2})}\right] \geq \left[\frac{[((1 - 2u_v)^{1/2} - 1)}{(1 - 3(1 - 2u_v)^{1/2})}\right]. \tag{9}$$

Thus, it is clear from Eq. (9) that

$$u_h \geq u_v (\text{ for an assigned value of } \sigma). \tag{10}$$

That is, for an assigned value of the ratio of central pressure to central energy-density $\sigma(\equiv \sigma_v)$, the compactness ratio of homogeneous density distribution, $u(\equiv u_h)$ should always be larger than or equal to the compactness ratio $u(\equiv u_v)$ of any regular solution*, compatible with the structure of general relativity. Or, in other words, for an assigned value of the compactness ratio, u, the *minimum* value of the ratio of central pressure to central energy-density, σ, corresponds to the homogeneous density sphere.

In the light of Eq. (10), let us assign the same value M for the total mass corresponding to various regular configurations in hydrostatic equilibrium. If we denote the density of the homogeneous sphere by E_h , we can write

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*Notice that this finding is also true for self-bound singular solutions because the ratio of (infinite) central pressure to density turns to be finite [examples of such solutions are well represented by Tolman's type V and VI solutions (Tolman 1939)]. Hence, the notion 'any regular' solution may be replaced by 'any static' solution [of course, with the requirement that the density decreases monotonically outwards from the centre].

$$E_h = 3M/(4\pi R_h{}^3) \tag{11}$$

where R_h denotes the radius of the homogeneous density sphere. If R_v represents the radius of any other regular sphere for the same mass M, the average density E_v of this configuration would correspond to

$$E_v = 3M/(4\pi R_v{}^3). \tag{12}$$

Eq. (10) indicates that $R_v \geq R_h$. By the use of Eqs. (11) and (12) we find that

$$E_v \leq E_h. \tag{13}$$

That is, for an assign value of σ the average energy-density of any regular configuration, E_v, should always be less than or equal to the density, E_h , of the homogeneous density sphere for the same mass M.

We point out that the regular configurations corresponding to a single exact solution, or EOS with a finite central and non-vanishing surface density, in fact, do not fulfill the definition of this 'actual' total mass, M, which appears in the exterior Schwarzschild solution [this definition asserts that, being the coordinate mass, the particular 'type' of density variation considered for it should remain 'unknown' to an external observe and this is possible only when the mass depends either upon the central density, or upon the surface density, and in any case, not upon both of them (Negi 2004b; Negi 2006). It follows, therefore, that the central density should be independent of the surface density or vice-versa, according to the density distribution assigned for the mass]. And the so called self-bound regular structures, in fact, violate this requirement as they correspond to a surface density which always depends upon the central density and vice-versa. Thus, the main findings of the study regarding this criterion can be summarized in the following manner:

(a) The gravitationally-bound regular configuration and self-bound singular structures, described by a single EOS or exact solution, fulfill the definition of the total mass, M, appears in the exterior Schwarzschild solution, hence the condition of hydrostatic equilibrium is naturally satisfied by these structures [this finding is fully consistent with the 'compatibility

criterion' , because for all (possible) values of σ, the condition $u \leq u_h$ is fully satisfied by these configurations].

(b) The self-bound regular configurations, described by a single EOS or exact solution, can not fulfill the definition of the total mass, M, appears in the exterior Schwarzschild solution, as a result, the state of hydrostatic equilibrium can not be satisfied by them [this finding is also fully consistent with the 'compatibility criterion' , because such configurations correspond to the condition, $u > u_h$ (for all possible values of σ)].

(c) The only regular configuration which can exist under the category (b) mentioned above is described by the homogeneous density distribution.

Note that the two-density or multiple-density models (that is, the structures governed by two or more EOSs assigned for different regions with appropriate matching conditions at the core-envelope boundaries) of both of the categories, (a) and (b) described above (such that the definition of the mass M mentioned above is appropriately satisfied) are quite possible, however, as we will show in the present paper that the fulfillment of the definition of the mass 'M' for any two-density model represents only a *necessary* condition for hydrostatic equilibrium, because the 'compatibility criterion' may not be satisfied by them. As we have noted earlier that the necessary condition for hydrostatic equilibrium (that is, the fulfillment of the definition of the mass M) put forward by the exterior Schwarzschild solution is also sufficient for a *single* EOS or exact solution assigned for the mass, because this fact is also supported by the 'compatibility criterion'. It follows therefore that the 'compatibility criterion' is capable of ensuring a sufficient and necessary condition for any structure (including two-density or multiple-density distribution) in the state of hydrostatic equilibrium.

To elaborate this statement more clearly, let us consider the core-envelope model discussed by Negi and Durgapal (2000). The core of this model is described by an EOS which belongs to the category '(b)' mentioned above, and the matching of various parameters at the core-envelope boundary is assured by characterizing an envelope which belongs to the category '(a)' EOS. That is, 'overall' the model describes a gravitationally bound two-

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density structure [of category 'a' mentioned above], such that the *necessary* condition for hydrostatic equilibrium put forward by exterior Schwarzschild solution at the surface of the configuration (that is, the mass, M, depends only upon the central density, meaning thereby that the definition of mass is appropriately satisfied), even then, the compatibility criterion for hydrostatic equilibrium (Negi & Durgapal 2001) turns out to be unsatisfied for this model (as shown under section 4 of the present study). Thus, it follows that the fulfillment of *necessary* condition for hydrostatic equilibrium at the surface, and the achievement of proper matching conditions at the core-envelope boundary are not *sufficient* to assure the condition of hydrostatic equilibrium for any *two-density* structure. However, the fulfillment of compatibility criterion alone could provides a *necessary* and *sufficient* condition for any regular configuration (including two-density structures) to be consistent with the state of hydrostatic equilibrium.

In order to verify this statement, we would re-investigate the core-envelope model put forward by Negi and Durgapal (2000), based upon the said compatibility criterion for hydrostatic equilibrium, such that for each (possible) assigned value of σ, the compactness ratio of the whole configuration, u, remains less than or equal to the compactness ratio, u_h, of the corresponding sphere of homogeneous density distribution. Such an investigation is possible, because we can re-adjust the boundary, r_b, of the core-envelope model in such a manner that for an assigned value of σ, the 'average density', E_{av}(say), of the whole configuration always remains less than or equal to the density, E_h, of the homogeneous density sphere for same mass M. Thus, this criterion should be fulfilled by any regular configuration specified by a single density distribution, a core-envelope model, a core-mantle-envelope model, or any other complicated distribution of matter composed of various regions inside the configuration, in order to fulfill the state of hydrostatic equilibrium. This statement is verified on the basis of dynamical stability of some regular configurations, consistent with the compatibility criterion, in the following sections, and the results which are summarized as Theorem 2 and its subsequent corollaries in the following section, may be stated in the general form as the following theorem.

Theorem 1: The necessary and sufficient condition for hydrostatic equilibrium of any static[†] and spherical configuration is that for an assigned value of the ratio of central pressure to central energy-density, the compactness ratio $u(\equiv M/R)$ of the said configuration should not exceed the compactness ratio u_h of the corresponding sphere of homogeneous density distribution.

3. Necessary and sufficient condition for hydrostatic equilibrium and dynamical stability of regular configuration

The *absolute* values are obtained by using a 'compressible' sphere of homogeneous energy-density (Negi 2004a), such that the following relation holds good for a constant Γ_1

$$\frac{\Gamma_1 P}{P + E} = \frac{dP}{dE}.$$

And the adiabatic speed of sound, $v = \sqrt{(dP/dE)}$, becomes finite inside this configuration for a finite (constant) Γ_1. In order to satisfy the extreme case of causality condition $v = c = 1$ at the centre of this sphere, we obtain $(P_0/E_0) \cong 0.6271$, which correspond to a u value $\cong 0.3406$, and the (critical) constant $\Gamma_1 = (\Gamma_1)_0 \cong 2.5946$ respectively, for the dynamically stable configuration. This value of $u(\cong 0.3406)$ represents an absolute upper bound, consistent with causality and dynamical stability, since it follows from the compatibility criterion that for this maximum value of u, the corresponding value of (P_0/E_0) of any regular configuration can not be less than 0.6271. Now, this result may be generalized for the sequences, composed of NS models such that every member of this sequence satisfies $(dP/dE)_0 = 1$ (here and elsewhere in the paper, the subscript '0' represent the value of the corresponding quantity at the centre), in the following manner that the maximum stable

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[†]the notion 'any static' instead of 'any regular' is used here in the general sense, since the dynamical stability of singular solutions (which may also satisfy the compatibility criterion) does not correspond to any solution.

value of u (corresponding to the case of first maxima among masses in the $M - R$ relation) and the corresponding central value of local $(\Gamma_1)[= (\Gamma_1)_0]$ of such sequences must satisfy the inequalities $u_{\max} \leq 0.3406$, and $(\Gamma_1)_0 \leq 2.5946$ respectively, in order to ensure the necessary and sufficient condition for dynamical stability of a mass. Since these absolute values are obtained by using the 'trial function', $\xi = re^{\nu/4}$, which is able to provide the necessary and sufficient condition for dynamical stability [‡] in the variational method (Chandrasekhar 1964a, b).

The equilibrium sequences of the type mentioned here, in fact, are widely discussed in the literature, but not on the basis of compatibility criterion. The core-envelope models presented by Negi & Durgapal (2000) also represent an equilibrium sequence of this type. We re-constructed this sequence on the basis of compatibility criterion for the first time (for other example of such a sequence, consistent with the compatibility criterion, see, e.g. Negi 2005), and it is seen that the maximum stable value of compactness ratio and the corresponding central value of local $(\Gamma_1)[= (\Gamma_1)_0]$ of such sequences are found fully consistent with those of the values obtained mentioned above (section 4; see also Negi 2005). It follows, therefore, that the $M - R$ relation (or the mass-central density relation) provides the necessary and sufficient condition for dynamical stability of a mass only when the compatibility criterion

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[‡]although, the variational method gives only a sufficient condition for the dynamical stability of a mass, since the results depend somewhat on the choice of a particular 'trial function'. However, we have shown that the choice of the particular trial function, $\xi = re^{\nu/4}$, is capable of providing the most rigorous results among the various trial functions (Negi & Durgapal 1999b; Negi 2004a), and because of this reason it could provide the necessary and sufficient condition for dynamical stability. It would not be out of place here to point out that the variational method gives precise results only for the configurations which correspond to a smooth variation of density from centre to the surface (see, e.g. Bardeen et. al 1966), and for this reason we did not consider this method for analyzing the stability of core-envelope models considered here.

for the equilibrium configuration is also satisfied. Or, in other words, the compatibility criterion is able to provide the necessary and sufficient condition for hydrostatic equilibrium of regular configurations.

In order to verify the last claim, irrespective of the particular type of core-envelope models considered in the present study, we would further consider (section 5) the dynamical stability of pure polytropic configurations ($P = K\rho^{\Gamma_1}$) on the basis of variational method for the choice of trial function $\xi = re^{\nu/4}$, as well as the mass-central density relation for some assigned values of constant Γ_1, since the polytropic configurations appropriately satisfy the 'compatibility criterion' (Negi & Durgapal 2001).

The results which would follow from the study of sections 4 and 5 respectively, may be summarized as theorem 2 and its subsequent corollaries in the following form

Theorem 2: The mass-radius (or, mass-central density) relation for an equilibrium sequence of regular configurations provides the necessary and sufficient condition for dynamical stability only when the equilibrium sequence it-self satisfies the necessary and sufficient condition of hydrostatic equilibrium (theorem 1).

Corollary 1 to theorem 2: If an equilibrium sequence, composed of neutron star models in such a manner that every member of this sequence satisfies the extreme causality condition $v = c = 1$ at the centre, then the maximum value of compactness ratio (corresponding to the case of first maxima among masses in the $M - R$ relation) and the corresponding central value of local adiabatic index $(\Gamma_1)_0$ are constrained by the inequalities, $u_{\max} \leq 0.3406$ and $(\Gamma_1)_0 \leq 2.5946$ respectively.

Corollary 2 to theorem 2: For regular configurations, corresponding to enough smooth density variations such that the variational method could be used, the variational method could provide the necessary and sufficient condition of dynamical stability just for the choice of a particular trial function $\xi = re^{\nu/4}$.

4. Hydrostatic equilibrium and dynamical stability of core-envelope models

The metric for spherically symmetric and static configurations can be written in the following form

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2, \tag{14}$$

where ν and λ are functions of r alone. Recalling that we are using 'geometrized units', the Oppenheimer-Volkoff (O-V) equations (Oppenheimer & Volkoff 1939), resulting from Einstein's field equations, for systems with isotropic pressure P and energy-density E can be written as

$$P' = -(P + E)[4\pi P r^3 + m]/r(r - 2m) \tag{15}$$

$$\nu'/2 = -P'/(P + E) \tag{16}$$

$$m'(r) = 4\pi E r^2 \,; \tag{17}$$

where $m(r)$ is the mass contained within the radius r, and the prime denotes radial derivative.

The core-envelope model (Negi & Durgapal 2000) consists of a core with most stiff EOS in the region $0 \le r \le b$, and an envelope with a polytropic EOS in the region $b \le r \le R$, given as

(i) The core: $0 \le r \le b$

For the models of neutron stars considered here, we have chosen the core of most stiff material as

$$P = (E - E_s) \tag{18}$$

where E_s is the value of density at the surface of the configuration, where pressure vanishes.

(ii) The envelope: $b \le r \le R$

The envelope of this model is given by the equation of state

$$P = K\rho^{\Gamma_1} \tag{19}$$

or

$$(E - \rho) = P/(\Gamma_1 - 1).$$

where K is a constant to be worked out by the matching of various variables at the core-envelope boundary and ρ and Γ_1 represent respectively, the rest-mass density and the (constant) adiabatic index as defined earlier.

At the boundary, $r = b$, the continuity of $P(= P_b), E(= E_b)$, and $r(= r_b)$ require

$$K = P_b/(E_b - [P_b/(\Gamma_1 - 1)])^{\Gamma_1} \tag{20}$$

where Γ_1 is given by (see, e.g., Tooper 1965)

$$\Gamma_1 = [(P + E)/P](dP/dE).$$

The continuity of (dP/dE), at the boundary gives

$$\Gamma_1 = 1 + (E_b/P_b). \tag{21}$$

Thus, the continuity of P, E, ν, λ, and (dP/dE) at the core-envelope boundary is ensured, for the static and spherically symmetric configuration.

The coupled Eqs. (15), (16), (17), are solved along with Eqs.(18) and (19) for the boundary conditions (20) and (21) [at the core-envelope boundary, $r = b$], and the boundary conditions, $P = E = 0$, $m(r = R) = M$, $e^\nu = e^{-\lambda} = (1 - 2M/R) = (1 - 2u)$ at the external boundary, $r = R$.

For the sake of numerical simplification, we assign the central density, $E_0 = 1$. It is seen that the degree of softness of the envelope is restricted by the inequality, $(P_b/E_b) \geq 0.014$. For the minimum value of $(P_b/E_b) \cong 0.014$, we obtain various quantities, such as, core mass, M_b, core radius, r_b , density at the core-envelope boundary, E_b , total mass, M, and the corresponding radius, R, of the configuration in dimensionless form. Some of these quantities are shown in Table 1 for various assigned values of the central pressure to density ratio, (P_0/E_0).

To determine the stability of the models given in Table 1, we need to draw the mass-radius diagram for the structures. For this purpose, we have normalized the boundary

density, $E_b = 2 \times 10^{14}$ g cm^{-3}, and obtained the mass-radius diagram as shown in Fig. 1 of Negi and Durgapal (2000) [Notice that the value of E_b chosen in this way (and hence also the mass and the radius obtained in conventional units as shown in Fig. 1) is purely arbitrary. These values have nothing to do with the actual maximum mass and the corresponding radius of the stable neutron star obtained in the present paper.]. The maximum stable value of u of the whole configuration is obtained as 0.3574. For this maximum value of u, the binding energy per baryon, $\alpha_r [\equiv (M_r - M)/M_r$, where M_r is the rest-mass (Zeldovich & Novikov 1978) of the configuration] also approaches maximum ($\cong 0.2441$) as shown in Table 1. Although, the corresponding $P_0/E_0(\simeq 0.704)$ value (or $(\Gamma_1)_0 \simeq 2.4204$ value) is consistent with the corresponding absolute value (P_0/E_0 is larger than 0.6271 or $(\Gamma_1)_0$ is less than 2.5946), the configuration is not consistent with the corollary 1 of theorem 2, since the maximum value of $u \simeq 0.3574$ is inconsistent with the absolute upper bound on $u \cong 0.3406$. It follows, therefore, that the $M - R$ relation does not provide the necessary and sufficient condition for dynamical stability of equilibrium masses, since these models are not consistent with the compatibility criterion, which is evident from Table 1. As the first column of Table 1 corresponds to compactness ratio, u_h, of homogeneous density sphere as calculated from Eqs. (1) and (2) for various assigned values of σ shown in Table 1. Column seventh of this table represents the compactness ratio u of the whole configuration for the same values of σ. Comparing column one and seventh, we find that for each assigned value of $\sigma, u > u_h$, meaning thereby that the model is inconsistent with the compatibility criterion.

To make the model consistent with the structure of general relativity, we re-investigate this model based upon the compatibility criterion (section 2) by solving the coupled Eqs. (15), (16), and (17) together with Eqs. (18) and (19) for the boundary conditions (20) and (21) respectively. For numerical simplicity, we assign the central energy-density of the configuration, $E_0 = 1$. The order of numerical precision is set precisely following the specific nature of EOSs for the core and envelope regions respectively. The ratio, (P_b/E_b), at the core-envelope boundary is so adjusted that for each and every (possible) assigned value of σ, the compactness ratio, u, of the whole configuration always turns out to be less than or

equal to the compactness ratio, u_h, of the homogeneous density sphere for same values of σ. The results obtained in this regard are shown in Table 2. It is seen that to meet the requirement set up by compatibility criterion, the minimum value of (P_b/E_b) reaches about 2.9201×10^{-1}.

To investigate the stability of the models which are now compatible with the structure of general relativity and causality (Table 2), we draw the mass-radius diagram for the models by normalizing the boundary density, $E_b = 2 \times 10^{14} \mathrm{g\,cm^{-3}}$, as shown in Fig. 1 [notice that the use of normalizing density, E_b, as mentioned in the previous case also, is purely arbitrary and its purpose is only to determine the maximum value of u upto which the structures remain pulsationally stable]. The *first* maxima in mass, among the equilibrium sequences of masses, is reached when the ratio of central pressure to central energy-density, $\sigma[\equiv (P_0/E_0)]$, approaches to a value about 0.6285. The binding-energy per baryon, $\alpha_r[\equiv (M_r - M)/M_r\,]$ also approaches to its *first* maxima for this maximum stable value of mass. The corresponding maximum stable value of compactness ratio, u, is obtained as 0.3389 (Table 2). Thus, the structure remains pulsationally stable upto a u value as large as 0.3389, that is, $u \le 0.3389$ [notice that both, the upper bound on maximum value of $u < 0.3406$, and $(\Gamma_1)_0 < 2.5946$ are fully consistent with the corresponding absolute upper bounds]. Thus, the corollary 1 of theorem 2 is fully satisfied for this case. It follows, therefore, that the $M - R$ relation provides the necessary and sufficient condition for dynamical stability of equilibrium masses, since these models are fully consistent with the compatibility criterion. This is evident from the comparison of column one and column seventh of Table 2 which indicates that for each value of (P_0/E_0), the compactness ratio of the whole configuration, u, is always less than u_h, the compactness ratio of the corresponding homogeneous density sphere. Obviously, the upper bound on $u \le 0.3389$ obtained here by using the compatibility criterion, turns out to be much stronger than the upper bounds on this parameter obtained by Lindblom (1984) and Haensel et al (1999).

5. Dynamical stability of polytropic configurations

The dynamical stability of polytropic configurations $(P = K\rho^{\Gamma_1})$ was first investigated by Tooper (1965) for polytropic index $3 \le n \le 1 (\Gamma_1 = 1 + 1/n)$ by using the variational method which states that a *sufficient* condition for the dynamical stability of a mass is that the right-hand side of the following equation

$$\omega^2 \int_0^R e^{(3\lambda-\nu)/2}(P+E)r^2\xi^2 dr =$$

$$4 \int_0^R e^{(\lambda+\nu)/2} rP'\xi^2 dr$$

$$+ \int_0^R e^{(\lambda+3\nu)/2}[\gamma P/r^2](r^2 e^{-\nu/2}\xi)'^2 dr$$

$$- \int_0^R e^{(\lambda+\nu/2)}[P'^2/(P+E)]r^2\xi^2 dr$$

$$+8\pi \int_0^R e^{(3\lambda+\nu)/2}P(P+E)r^2\xi^2 dr. \tag{22}$$

vanishes for some chosen "trial function" ξ which satisfies the boundary conditions

$$\xi = 0 \quad \text{at} \quad r = 0, \tag{23}$$

and

$$\delta P = -\xi P' - \gamma P e^{\nu/2}[(r^2 e^{-\nu/2}\xi)'/r^2]$$

$$= 0 \quad \text{at} \quad r = R, \tag{24}$$

where ω is the angular frequency of pulsation, R is the size of the configuration, and δP is the 'Lagrangian displacement in pressure'. The prime denotes radial derivative, and the quantity $\gamma = [(P+E)/P](dP/dE) = \Gamma_1$ (constant) for the polytropic configurations considered here. Tooper (1965) used the trial function of the form $\xi = b_1 r(1 + a_1 r^2 + a_2 r^4 + a_3 r^6 + ...)e^{\nu/2}$, where $a_1, a_2, a_3, ...$ are adjustable constants, in Eq.(22) and showed that for $3 < n \le 1$, the first maxima among the masses in the mass-central density (or, mass-radius) relation approaches at the same value of central pressure to central rest-density ratio (P_0/ρ_0) where the squared frequency of pulsation, ω^2, also becomes zero.

In order to verify these results in view of the discussion of section 3, we choose the trial function $\xi = re^{\nu/4}$ and employ a fourth-order Runge-Kutta method to solve Eq.(22). The results of this iteration are presented in Tables 3-4 for the polytropic index $n = 1$ and 1.5 respectively. It is seen that the first maxima among masses in the mass-centre density relation is reached for the same value of (P_0/ρ_0) where ω^2 also approaches zero. This finding is in perfect agreement with those of the Tooper (1965) and together with theorem 2 verifies further that just the choice of the trial function $\xi = re^{\nu/4}$ is capable of providing the necessary and sufficient condition for dynamical stability of masses.

6. Results and conclusions

We have re-investigated the core-envelope model with stiffest equation of state [speed of sound equal to that of light] in the core and a polytropic equation with constant adiabatic index $\Gamma_1 = [\mathrm{d}lnP/\mathrm{d}ln\rho]$ in the envelope, based upon the criterion obtained by Negi and Durgapal (2001). We find that the condition of hydrostatic equilibrium is assured only when the minimum ratio of (P_b/E_b) at the core-envelope boundary reaches about 2.9201×10^{-1} [that is, when the value of the adiabatic index, Γ_1 at the core-envelope boundary reaches around 4.4246 as compared to the previous case of $\Gamma_1 \cong 72.4286$]. Under this condition, the pressure, density, both of the metric parameters including their first derivatives, and the speed of sound are continuous at the core-envelope boundary and at the surface. The mass-radius diagram indicates that the configuration remains dynamically stable upto a u value as large as 0.3389. The corresponding central value of the local $(\Gamma_1)_0$ is obtained as 2.5911. These values are fully consistent with those of the *absolute* values, $u_{max} \cong 0.3406$, and $(\Gamma_1)_{0,max} \cong 2.5946$, compatible with the structure of general relativity, causality, and dynamical stability, obtained by using a causal configuration of homogeneous energy-density (Negi 2004a). Not just for the particular model considered in the present study, we have also found that the maximum value of u and the corresponding value of $(\Gamma_1)_0$ upto which the configurations remain pulsationally stable can not exceed the values of $u \simeq 0.34$ and

$(\Gamma_1)_0 \simeq 2.5128$ respectively, if the 'compatibility criterion' is followed and the envelope of the present model is replaced by the polytropic EOS $(4/3) \leq \mathrm{d}lnP/\mathrm{d}ln\rho \leq 2$ (Negi 2005).

In addition to the study of two-density models considered in the present study (sec.4), the dynamical stability of the polytropic configurations (sec.5) explicitly shows that the compatibility criterion (Negi & Durgapal 2001) alone is capable of providing a necessary and sufficient condition for any regular configuration to be consistent with the structure of general relativity [however, the study of $M - R$ relation of such sequences (corresponding to two-density models with $v = c = 1$ at the centre), consistent with the definition of actual mass (Negi 2004 b; Negi 2006) and the 'compatibility criterion' (Negi & Durgapal 2001) in the near future will finally settle down this issue].

The two-density structures are dynamically stable and gravitationally bound even for the value of compactness ratio, $u \geq (1/3)$, thus giving a suitable model for studying the Ultra-compact Objects [UCOs] discussed in the literature (see, e. g., Negi & Durgapal 1999a, b; 2000; and references therein). The present type of studies may also find application to test various models of NSs based upon EOSs of dense nuclear matter, and the models of relativistic stellar objects like - star clusters.

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Table 1: Properties of the causal core-envelope models, with a core given by the most stiff EOS, $(dP/dE) = 1$, and the envelope is characterized by the polytropic EOS, $(\mathrm{d}lnP/\mathrm{d}ln\rho) = \Gamma_1$, such that, all the parameters, P, E, ν, λ, and the speed of sound, $(dP/dE)^{1/2}$, are continuous at the core-envelope boundary, r_b, and the models satisfy the necessary (but not sufficient) condition for hydrostatic equilibrium. The maximum value of $u[\equiv (M/R) \cong 0.3574]$ for the structure is obtained [Fig. 1 of Negi & Durgapal (2000)], when the minimum value of the ratio of pressure to density at the core-envelope boundary, (P_b/E_b), reaches about 0.014. The maximum value of the binding-energy per baryon, $\alpha_r[\equiv (M_r - M)/M_r$, where M_r is the rest mass of the configuration] $\cong 0.2441$, also occurs for the maximum stable value of u. The subscript '0' and 'b' represent, the values of respective quantities at the centre, and at the core-envelope boundary. z_R stands for the surface redshift. The calculations are performed for an assigned value of the central energy-density, $E_0 = 1$. Various values shown in the table are round off at the fourth decimal place. The slanted values represent the limiting case upto which the structure remains dynamically stable. However, the model do not satisfy the necessary and sufficient condition for hydrostatic equilibrium, since for each assigned value of (P_0/E_0), the compactness ratio, u, of the whole configuration always corresponds to a value *larger* than that of the compactness ratio, u_h, of the homogeneous density distribution (that is $u > u_h$). Furthermore, the $M - R$ relation does not provide the necessary and sufficient condition for dynamical stability of equilibrium configurations, since the maximum stable value of $u \simeq 0.3574$ exceeds the limiting value ($u \cong 0.3406$) for the corresponding centre value of $(\Gamma_1)_0 \simeq 2.4204$ (which is, however, consistent with the corresponding absolute upper bound ≤ 2.5946).

TABLES

u_h	P_0/E_0	$(\Gamma_1)_0$	r_b	E_b	R	u	α_r	z_R	z_0
0.1527	0.1110	10.009	0.1886	0.9017	0.2012	0.1558	0.1001	0.2052	0.3473
0.1901	0.1562	7.4020	0.2181	0.8559	0.2282	0.1946	0.1290	0.2795	0.4977
0.2195	0.2012	5.9702	0.2410	0.8101	0.2489	0.2243	0.1512	0.3467	0.6515
0.2478	0.2564	4.9002	0.2639	0.7542	0.2712	0.2550	0.1751	0.4287	0.8573
0.2696	0.3100	4.2258	0.2832	0.6998	0.2887	0.2763	0.1917	0.4949	1.0598
0.2975	0.4000	3.5000	0.3127	0.6086	0.3176	0.3058	0.2152	0.0648	1.4513
0.3213	0.5070	2.9724	0.3477	0.5001	0.3522	0.3304	0.2334	0.7168	2.0020
0.3315	0.5661	2.7665	0.3690	0.4401	0.3733	0.3405	0.2395	0.7703	2.3633
0.3369	0.6010	2.6639	0.3829	0.4047	0.3870	0.3455	0.2419	0.7988	2.6033
0.3424	0.6410	2.5601	0.4003	0.3642	0.4043	0.3504	0.2434	0.8282	2.9084
0.3501	*0.7040*	*2.4204*	*0.4329*	*0.3002*	*0.4376*	*0.3574*	*0.2441*	*0.8724*	*3.4925*
0.3504	0.7070	2.4144	0.4347	0.2971	0.4395	0.3577	0.2441	0.8744	3.5242
0.3513	0.7151	2.3984	0.4396	0.2890	0.4443	0.3583	0.2436	0.8783	3.6085
0.3523	0.7238	2.3816	0.4450	0.2801	0.4497	0.3590	0.2430	0.8833	3.7055

Table 2: Properties of the causal core-envelope models, as discussed in the present paper, with a core given by the most stiff EOS, $(dP/dE) = 1$, and the envelope is characterized by the polytropic EOS, $(\mathrm{d}lnP/\mathrm{d}ln\rho) = \Gamma_1$, such that, all the parameters, P, E, ν, λ, and the speed of sound, $(dP/dE)^{1/2}$, are continuous at the core-envelope boundary, r_b. The maximum value of $u[\equiv (M/R) \cong 0.3389]$ for the structure is obtained (Fig. 1), when the minimum value of the ratio of pressure to density at the core-envelope boundary, (P_b/E_b), reaches about 2.9201×10^{-1}. The *first* maxima among the values of the binding-energy per baryon, $\alpha_r[\equiv (M_r - M)/M_r$, where M_r is the rest mass of the configuration] also occurs for the maximum stable value of u. The calculations are performed for an assigned value of the central energy-density, $E_0 = 1$. Except (P_0/E_0) and r_b, all other values are round off at the fourth decimal place. The subscript '0' and 'b' represent, the values of respective quantities at the centre, and at the core-envelope boundary. z_R stands for the surface redshift. The slanted values represent the limiting case upto which the structure remains dynamically stable. The model is fully compatible with the structure of general relativity, as it is seen that for each assigned value of (P_0/E_0), the compactness ratio, u, of the whole configuration always corresponds to a value *less than or equal to* that of the compactness ratio, u_h, of the homogeneous density distribution (that is $u \leq u_h$). The maximum stable value of $u \simeq 0.3389(< 0.3406)$ and the corresponding central value of $(\Gamma_1)_0 \simeq 2.5911(< 2.5946)$ indicate that the $M-R$ relation provides the necessary and sufficient condition for dynamical stability of equilibrium configurations.

u_h	P_0/E_0	$(\Gamma_1)_0$	r_b	E_b	R	u	α_r	z_R	z_0
(10^0)	(10^0)	(10^0)	(10^{-3})	(10^0)	(10^0)	(10^0)	(10^0)	(10^0)	(10^0)
0.2628	0.29202	4.4244	2.68	0.9999	0.3117	0.2628	0.1698	0.4520	0.9615
0.2652	0.29800	4.3557	41.25	0.9915	0.3132	0.2649	0.1714	0.4584	0.9831
0.2846	0.35452	3.8207	128.53	0.9117	0.3275	0.2837	0.1870	0.5205	1.2026
0.2976	0.39993	3.5005	166.04	0.8476	0.3400	0.2965	0.1984	0.5676	1.3943
0.3072	0.43923	3.2767	191.90	0.7920	0.3509	0.3063	0.2069	0.6066	1.5738
0.3112	0.45699	3.1882	202.42	0.7670	0.3564	0.3102	0.2105	0.6232	1.6588
0.3188	0.49397	3.0244	222.83	0.7147	0.3678	0.3178	0.2170	0.6568	1.8467
0.3261	0.53331	2.8751	243.11	0.6592	0.3801	0.3252	0.2224	0.6913	2.0657
0.3274	0.54099	2.8485	246.97	0.6483	0.3830	0.3265	0.2235	0.6974	2.1101
0.3306	0.55999	2.7857	256.44	0.6215	0.3894	0.3296	0.2256	0.7130	2.2255
0.3355	0.59146	2.6907	272.04	0.5770	0.4020	0.3341	0.2289	0.7361	2.4265
0.3394	0.61902	2.6154	285.84	0.5381	0.4128	0.3380	0.2307	0.7568	2.6215
0.3407	*0.62850*	*2.5911*	*290.65*	*0.5247*	*0.4176*	*0.3389*	*0.2314*	*0.7618*	*2.6888*
0.3409	0.62936	2.5889	291.09	0.5235	0.4171	0.3394	0.2311	0.7642	2.6989
0.3411	0.63148	2.5836	292.18	0.5205	0.4182	0.3396	0.2313	0.7654	2.7146
0.3438	0.65148	2.5350	302.55	0.4922	0.4276	0.3419	0.2322	0.7783	2.8710
0.3497	0.69998	2.4286	329.29	0.4238	0.4533	0.3467	0.2328	0.8061	3.2991
0.3553	0.75294	2.3281	362.83	0.3490	0.4888	0.3505	0.2306	0.8290	3.8720

Table 3: Properties of the models characterized by the pure polytropic EOS, $(\mathrm{d}lnP/\mathrm{d}ln\rho) = \Gamma_1$, for $\Gamma_1 = 5/3(n = 1.5)$. The stability of the models is judged by the variational method for the choice of the trial function $\xi = re^{\nu/4}$ in Eq.(22), as well as the mass-central density relation [equivalent to dimensionless mass (M/M^*) vs. (P_0/E_0) or (P_0/ρ_0) ratio; where $M^* = (n+1)^{3/2}(P_0/\rho_0)^{n/2}/(4\pi\rho_0)^{1/2}$]. It is apparently seen that the configurations become dynamically unstable beyond the maximum mass where the squared angular frequency of pulsation, ω^2, also approaches zero. Thus, it follows that the mass-central density relation provides the necessary and sufficient condition for dynamical stability of equilibrium configurations. The maximum value of the binding-energy per baryon, $\alpha_r[\equiv (M_r - M)/M_r$, where M_r is the rest mass of the configuration] also occurs for the maximum value of mass. All values are round off at the fourth decimal place. The subscript '0' represents, the values of respective quantities at the centre. The slanted values represent the limiting case upto which the structure remains dynamically stable. The model is fully compatible with the structure of general relativity, as it is seen that for each assigned value of (P_0/E_0), the compactness ratio, u, of the whole configuration always corresponds to a value *less than* that of the compactness ratio, u_h, of the homogeneous density distribution (that is $u < u_h$).

P_0/E_0	P_0/ρ_0	M/M^*	α_r	ω^2/E_0	u	u_h
0.0465	0.0500	0.1907	0.0277	1.1085	0.0698	0.0783
0.0653	0.0724	0.2153	0.0343	0.8047	0.0904	0.1032
0.0805	0.0916	0.2272	0.0381	0.5734	0.1046	0.1213
0.1026	0.1213	0.2359	0.0414	0.2669	0.1221	0.1446
0.1158	0.1402	0.2378	0.0422	0.0979	0.1308	0.1571
0.1239	*0.1522*	*0.2380*	*0.0424*	*0.0000*	*0.1356*	0.1643
0.1300	0.1615	0.2377	0.0422	-0.0715	0.1391	0.1695
0.1523	0.1974	0.2346	0.0405	-0.3130	0.1501	0.1872
0.1798	0.2462	0.2273	0.0357	-0.5694	0.1604	0.2061
0.2696	0.4526	0.1924	0.0021	-1.0936	0.1750	0.2537
0.3279	0.6453	0.1684	-0.0324	-1.1804	0.1716	0.2760
0.3705	0.8340	0.1523	-0.0624	-1.1146	0.1635	0.2894

Table 4: Properties of the models characterized by the pure polytropic EOS, $(\mathrm{d}lnP/\mathrm{d}ln\rho) = \Gamma_1$, for $\Gamma_1 = 2(n = 1)$. The stability of the models is judged by the variational method for the choice of the trial function $\xi = re^{\nu/4}$ in Eq.(22), as well as the mass-central density relation [equivalent to dimensionless mass (M/M^*) vs. (P_0/E_0) or (P_0/ρ_0) ratio; where $M^* = (n+1)^{3/2}(P_0/\rho_0)^{n/2}/(4\pi\rho_0)^{1/2}$]. It is apparently seen that the configurations become dynamically unstable beyond the maximum mass where the squared angular frequency of pulsation, ω^2, also approaches zero. Thus, it follows that the mass-central density relation provides the necessary and sufficient condition for dynamical stability of equilibrium configurations. The maximum value of the binding-energy per baryon, $\alpha_r[\equiv (M_r - M)/M_r$, where M_r is the rest mass of the configuration] also occurs for the maximum value of mass. All values are round off at the fourth decimal place. The subscript '0' represents, the values of respective quantities at the centre. The slanted values represent the limiting case upto which the structure remains dynamically stable. The model is fully compatible with the structure of general relativity, as it is seen that for each assigned value of (P_0/E_0), the compactness ratio, u, of the whole configuration always corresponds to a value *less than* that of the compactness ratio, u_h, of the homogeneous density distribution (that is $u < u_h$).

P_0/E_0	P_0/ρ_0	M/M^*	α_r	ω^2/E_0	u	u_h
0.1114	0.1254	0.1752	0.0699	2.2225	0.1452	0.1530
0.1252	0.1431	0.1830	0.0744	1.9497	0.1559	0.1655
0.1579	0.1875	0.1958	0.0824	1.3440	0.1773	0.1913
0.2203	0.2826	0.2055	0.0895	0.3512	0.2077	0.2301
0.2309	0.3002	0.2059	0.0898	0.2030	0.2115	0.2356
0.2460	*0.3263*	*0.2060*	*0.0899*	*0.0000*	*0.2167*	0.2430
0.2548	0.3420	0.2058	0.0897	-0.1126	0.2194	0.2471
0.2800	0.3889	0.2045	0.0885	-0.4151	0.2265	0.2580
0.3266	0.4850	0.2000	0.0835	-0.8980	0.2365	0.2755
0.3948	0.6523	0.1902	0.0704	-1.4356	0.2458	0.2961
0.4430	0.7952	0.1820	0.0576	-1.7007	0.2486	0.3080
0.4917	0.9674	0.1733	0.0418	-1.8762	0.2490	0.3184

FIGURES

Fig. 1.   Mass-Radius diagram of the model corresponding to Table 2, for an assigned value of $E = E_b = 2 \times 10^{14}$ g cm^{-3} at the core-envelope boundary r_b, such that the compactness ratio, u, of the whole configuration always turns out less than or equal to the compactness ratio , u_h, of homogeneous density sphere. This requirement is fulfilled only when the ratio of pressure to density (P_b/E_b) at r_b reaches to a minimum value about 2.9201×10^{-1}. The pressure, energy-density, ν, λ, and the speed of sound, $(dP/dE)^{1/2}$ are continuous at the core-envelope boundary.